

Vertex Operators of Super Wilson Loops

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Abstract

We study the supersymmetric Wilson loop as introduced by Caron-Huot, which attaches to lightlike polygons certain edge and vertex operators, whose shape is determined by supersymmetry constraints. We state explicit formulas for the vertex operators to all orders in the Grassmann expansion, thus filling a gap in the literature. This is achieved by deriving a recursion formula out of the supersymmetry constraints.

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I. INTRODUCTION

Gluon scattering amplitudes have been known to be dual to Wilson loops along lightlike polygons. While first shown at strong coupling (Ref. 1) through the famous AdS/CFT duality introduced in Ref. 2, this result has later been verified at weak coupling (Refs. 3 and 4). For a review, consult Ref. 5.

Recently, a similar duality (at weak coupling) between the full superamplitude of $\mathcal{N} = 4$ super Yang-Mills theory and a supersymmetric extension of the Wilson loop has been claimed, of which two variants appeared almost simultaneously. The first approach by Mason and Skinner (Ref. 6) originates in momentum twistor space and translates into the integral over a superconnection in spacetime. The second is due to Caron-Huot (Ref. 7) and attaches to lightlike polygons certain edge and vertex operators, whose shape is determined by supersymmetry constraints. At the classical level, both approaches are identical only on-shell (Ref. 8). Belitsky showed in Ref. 9 that the conjectured duality with superamplitudes indeed holds, however only upon subtracting an anomalous contribution from the super Wilson loop.

The operators in the Caron-Huot approach depend on momentum supertwistors. While the edge operators are wellknown, explicit formulas for the vertex operators have been available in the literature only up to fourth order in the Grassmann expansion. The aim of this article is to fill this gap. We state explicit formulas for the vertex operators up to maximum order. This is achieved by deriving a recursion formula out of the supersymmetry constraints.

To fix notation, we let W_n denote the super Wilson loop and \mathcal{E}_i and $\mathcal{V}_{i,i+1}$ the edge and vertex operators, respectively, which depend on the (odd) momentum supertwistors η_i^A and η_{i+1}^A . At zeroth order, the ordinary Wilson loop should be recovered, thus leading to the ansatz $\mathcal{E}_i = p_i \cdot A + \mathcal{O}(\eta)$ and $\mathcal{V}_{i,i+1} = 1 + \mathcal{O}(\eta)$. The supersymmetry constraints are such that $\mathcal{Q}_A^\alpha W_n = 0$ is to vanish, where the $\mathcal{Q}_A^\alpha = q_A^\alpha + c_0 \sum_i \lambda_i^\alpha \frac{\partial}{\partial \eta_i^A}$ act on the fields as well as the momentum supertwistors. This is achieved if the edges and vertices transform by an infinitesimal super gauge transformation

$$\mathcal{Q}_A^\alpha \mathcal{E}_i = \frac{1}{g} (\partial_t - ig [\mathcal{E}_i, \cdot]) X_{iA}^\alpha \quad (1a)$$

$$\mathcal{Q}_A^\alpha \mathcal{V}_{i,i+1} = i X_{i+1A}^\alpha \mathcal{V}_{i,i+1} - i \mathcal{V}_{i,i+1} X_{iA}^\alpha \quad (1b)$$

Here, and in the following, we adopt the conventions of Ref. 10.

II. EDGE OPERATORS

The edge operators are computed as sketched in Ref. 7. One finds the following solution of (1a), making use of the Euler-Lagrange equations.

$$\begin{aligned}\mathcal{E}_i = & \frac{1}{2}\lambda_{i\beta}\tilde{\lambda}_{i\dot{\beta}}A^{\beta\dot{\beta}} + \frac{i}{c_0}\tilde{\lambda}_{i\dot{\beta}}\tilde{\psi}_A^{\dot{\beta}}\eta_i^A - \frac{i\sqrt{2}}{2c_0^2}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\gamma}\bar{\phi}_{AB}}{\langle i, i-1 \rangle}\eta_i^A\eta_i^B \\ & + \frac{1}{3c_0^3}\varepsilon_{ABCD}\frac{\lambda_{(i-1)\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\gamma}\psi^{\xi A}}{\langle i, i-1 \rangle^2}\eta_i^B\eta_i^C\eta_i^D \\ & + \frac{i}{24c_0^4}\varepsilon_{ABCD}\frac{\lambda_{(i-1)\gamma}\lambda_{(i-1)\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\beta}D^{\dot{\beta}\beta}F^{\gamma\xi}}{\langle i, i-1 \rangle^3}\eta_i^A\eta_i^B\eta_i^C\eta_i^D\end{aligned}$$

with

$$\begin{aligned}X_{iA}^\alpha := & \frac{g\lambda_{i-1}^\alpha}{c_0\langle i, i-1 \rangle}\left(-2i\sqrt{2}\bar{\phi}_{AB}\eta_i^B + \varepsilon_{ABCD}\frac{2\lambda_{(i-1)\gamma}\psi^{\gamma B}}{c_0\langle i, i-1 \rangle}\eta_i^C\eta_i^D\right. \\ & \left. + \frac{i}{3c_0^2}\varepsilon_{ABCD}\frac{\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}F^{\gamma\beta}}{\langle i, i-1 \rangle^2}\eta_i^B\eta_i^C\eta_i^D\right)\end{aligned}$$

III. A RECURSION FORMULA

We expand

$$\mathcal{V}_{i,i+1} = \sum_{k=0}^4 \sum_{l=0}^4 V_{A_1\dots A_k B_1\dots B_l} \eta_i^{A_1} \dots \eta_i^{A_k} \eta_{i+1}^{B_1} \dots \eta_{i+1}^{B_l} \quad (2)$$

and, similarly, denote the coefficients of X_{iA}^α by

$$X_{iA}^\alpha = X_{iA}^{\alpha(1)} + X_{iA}^{\alpha(2)} + X_{iA}^{\alpha(3)} = X_{iAA_1}^{\alpha(1)}\eta_i^{A_1} + X_{iAA_1A_2}^{\alpha(2)}\eta_i^{A_1}\eta_i^{A_2} + X_{iAA_1A_2A_3}^{\alpha(3)}\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A_3}$$

Let $V_0 = 1$ (i.e. $\mathcal{V}_{i,i+1} = 1 + \mathcal{O}(\eta)$) and require that $\mathcal{V}_{i,i+1}$ only depends on the generators η_i and η_{i+1} . Then (1b) with X_{iA}^α as above has the following unique solution: All coefficients $V_{B_1,\dots,B_d} = 0$ for $d > 0$ (i.e. all "pure η_{i+1} -terms") vanish and the remaining coefficients are

successively determined by the following recursion formula.

$$\begin{aligned}
& V_{A A_1 \dots A_k B_1 \dots B_l} \\
&= \frac{(-1)^{d+1} \lambda_{(i+1)\alpha}}{(k+1) c_0 \langle i+1, i \rangle} \left(-q_A^\alpha (V_{A_1 \dots A_k B_1 \dots B_l}) + i X_{(i+1)AB_l}^{\alpha(1)} V_{A_1 \dots A_k B_1 \dots B_{l-1}} \right. \\
&\quad + i X_{(i+1)AB_{l-1}B_l}^{\alpha(2)} V_{A_1 \dots A_k B_1 \dots B_{l-2}} + i X_{(i+1)AB_{l-2}B_{l-1}B_l}^{\alpha(3)} V_{A_1 \dots A_k B_1 \dots B_{l-3}} \\
&\quad - i(-1)^l V_{A_1 \dots A_{k-1} B_1 \dots B_l} X_{iAA_k}^{\alpha(1)} - i(-1)^d V_{A_1 \dots A_{k-2} B_1 \dots B_l} X_{iAA_{k-1}A_k}^{\alpha(2)} \\
&\quad \left. - i(-1)^l V_{A_1 \dots A_{k-3} B_1 \dots B_l} X_{iAA_{k-2}A_{k-1}A_k}^{\alpha(3)} \right)
\end{aligned}$$

where $d = k + l$.

Proof. For calculations, it is easier to work with an expansion where the generators η_i and η_{i+1} can stand in any order:

$$\mathcal{V}_{i,i+1} = \sum_{d=0}^8 C_{B_1 \dots B_d}^{j_1 \dots j_d} \eta_{j_1}^{B_1} \dots \eta_{j_d}^{B_d}, \quad V_{A_1 \dots A_k B_1 \dots B_l} = \binom{k+l}{k} C_{A_1 \dots A_k B_1 \dots B_l}^{i \dots i \ i+1 \dots i+1}$$

with $j_i \in \{i, i+1\}$. Now, applying from the left a fixed $\frac{\partial}{\partial \eta_k^\alpha}$ in the C -expansion kills the corresponding η terms which can occur at every position, thus giving a symmetry factor of d and a sign such that

$$\mathcal{Q}_A^\alpha(\mathcal{V}_{i,i+1}) = \sum_{d=0}^8 \left(q_A^\alpha (C_{B_1 \dots B_d}^{j_1 \dots j_d}) + c_0(d+1) (-1)^{|C_{AB_1 \dots B_d}^{kj_1 \dots j_d}|} \lambda_k^\alpha C_{AB_1 \dots B_d}^{kj_1 \dots j_d} \right) \eta_{j_1}^{B_1} \dots \eta_{j_d}^{B_d}$$

(1b) is thus equivalent to the recursion formula

$$\begin{aligned}
& c_0(d+1) (-1)^{|C_{AB_1 \dots B_d}^{kj_1 \dots j_d}|} \lambda_k^\alpha C_{AB_1 \dots B_d}^{kj_1 \dots j_d} \eta_{j_1}^{B_1} \dots \eta_{j_d}^{B_d} \\
&= -q_A^\alpha (C_{B_1 \dots B_d}^{j_1 \dots j_d}) \eta_{j_1}^{B_1} \dots \eta_{j_d}^{B_d} + i \sum_{k+l=d} (X_{i+1A}^\alpha |_{\eta^k} \mathcal{V}_{i,i+1} |_{\eta^l} - \mathcal{V}_{i,i+1} |_{\eta^k} X_{iA}^\alpha |_{\eta^l})
\end{aligned}$$

By induction, one shows that the coefficients are of parity $|C_{B_1 \dots B_d}^{j_1 \dots j_d}| \equiv_2 d$. Also by induction, we see that all coefficients $C_{B_1 \dots B_d}^{i+1 \dots i+1} = 0$ vanish: In the recursion formula so far established, we consider the case $j_1 = \dots j_d = i+1$ and multiply both sides with $\lambda_{i\alpha}$. Then only the left hand side with $k = i+1$ remains and

$$C_{AB_1 \dots B_d}^{i+1, i+1 \dots i+1} \eta_{i+1}^{B_1} \dots \eta_{i+1}^{B_d} = \frac{(-1)^{d+1} \lambda_{i\alpha}}{\langle i, i+1 \rangle c_0(d+1)} \left(-q_A^\alpha (C_{B_1 \dots B_d}^{i+1 \dots i+1}) \eta_{i+1}^{B_1} \dots \eta_{i+1}^{B_d} \right)$$

since $\lambda_{i\alpha} X_{i+1A}^\alpha = 0$ and $X_{iA}^\alpha = \mathcal{O}(\eta_i)$. For $d = 0$, the right hand side $\sim q_A^\alpha(1) = 0$ vanishes and thus $C_B^{i+1} = 0$. Take this as induction basis and assume that $C_{B_1 \dots B_d}^{i+1 \dots i+1} = 0$. The same recursion formula then implies that $C_{AB_1 \dots B_d}^{i+1, i+1 \dots i+1} = 0$.

Now, by multiplying both sides of the recursion formula with $\lambda_{(i+1)\alpha}$, only the left hand side with $k = i$ remains, and we yield

$$\begin{aligned} & C_{AB_1 \dots B_d}^{ij_1 \dots j_d} \eta_{j_1}^{B_1} \dots \eta_{j_d}^{B_d} \\ &= \frac{(-1)^{d+1} \lambda_{(i+1)\alpha}}{\langle i+1, i \rangle c_0(d+1)} \left(-q_A^\alpha (C_{B_1 \dots B_d}^{j_1 \dots j_d}) \eta_{j_1}^{B_1} \dots \eta_{j_d}^{B_d} + (i X_{i+1 A}^\alpha \mathcal{V}_{i,i+1} - i \mathcal{V}_{i,i+1} X_{iA}^\alpha) |_{\eta^d} \right) \end{aligned}$$

Writing the second term on the right hand side in the C -expansion and then translating everything back to the original expansion (2) using

$$C_{AB_1 \dots B_d}^{ij_1 \dots j_d} \eta_{j_1}^{B_1} \dots \eta_{j_d}^{B_d} |_{\eta_i^k \eta_{i+1}^l} = \frac{k+1}{d+1} V_{A A_1 \dots A_k B_1 \dots B_l} \eta_i^{A_1} \dots \eta_i^{A_k} \eta_{i+1}^{B_1} \dots \eta_{i+1}^{B_l}$$

the statement is finally obtained. \square

IV. VERTEX OPERATORS

By the recursion formula of the previous section, the coefficients of the vertex operators in the expansion (2) can be explicitly calculated. Up to order three, the result reads

$$\begin{aligned} \mathcal{V}_{i,i+1} &= 1 - \frac{\sqrt{2} g i_\pm \bar{\phi}_{A_1 A_2}}{c_0^2 i_- i_+} \eta_i^{A_1} \eta_i^{A_2} + \frac{2\sqrt{2} g \bar{\phi}_{A_1 B_1}}{c_0^2 i_+} \eta_i^{A_1} \eta_{i+1}^{B_1} \\ &+ \frac{2ig i_\pm (-i_- \lambda_{(i+1)\gamma} + i_+ \lambda_{(i-1)\gamma}) \psi^{\gamma C}}{3c_0^3 i_-^2 i_+^2} \varepsilon_{A_1 A_2 A_3 C} \eta_i^{A_1} \eta_i^{A_2} \eta_i^{A_3} \\ &+ \frac{2ig \lambda_{(i+1)\gamma} \psi^{\gamma C}}{c_0^3 i_+^2} \varepsilon_{A_1 A_2 B_1 C} \eta_i^{A_1} \eta_i^{A_2} \eta_{i+1}^{B_1} - \frac{2ig \lambda_{i\gamma} \psi^{\gamma C}}{c_0^3 i_+^2} \varepsilon_{A_1 B_1 B_2 C} \eta_i^{A_1} \eta_{i+1}^{B_1} \eta_{i+1}^{B_2} \\ &+ \mathcal{O}(\eta^4) \end{aligned}$$

with $i_- := \langle i, i-1 \rangle$, $i_+ := \langle i+1, i \rangle$ and $i_\pm := \langle i+1, i-1 \rangle$.

Fourth Order

The (non-vanishing) fourth order coefficients (2) of $\mathcal{V}_{i,i+1}$ are as follows.

$$\begin{aligned}
V_{A_1 A_2 A_3 A_4} &= \left(\frac{g i_{\pm} (i_{-}^2 \lambda_{(i+1)\beta} \lambda_{(i+1)\gamma} - i_{-} i_{+} \lambda_{(i-1)\beta} \lambda_{(i+1)\gamma} + i_{+}^2 \lambda_{(i-1)\beta} \lambda_{(i-1)\gamma}) F^{\beta\gamma}}{12 c_0^4 i_{-}^3 i_{+}^3} \right. \\
&\quad \left. + \frac{g^2 i_{\pm}^2 \bar{\phi}_{CD} \phi^{CD}}{12 c_0^4 i_{-}^2 i_{+}^2} \right) \varepsilon_{A_1 A_2 A_3 A_4} \\
V_{A_1 A_2 A_3 B_1} &= -\frac{g \lambda_{(i+1)\beta} \lambda_{(i+1)\gamma} F^{\beta\gamma}}{3 c_0^4 i_{+}^3} \varepsilon_{A_1 A_2 A_3 B_1} - \frac{4 g^2 i_{\pm}}{c_0^4 i_{-} i_{+}^2} \bar{\phi}_{A_1 B_1} \bar{\phi}_{A_2 A_3} \\
V_{A_1 A_2 B_1 B_2} &= \frac{g \lambda_{i\beta} \lambda_{(i+1)\gamma} F^{\beta\gamma}}{2 c_0^4 i_{+}^3} \varepsilon_{A_1 A_2 B_1 B_2} - \frac{g^2}{c_0^4 i_{+}^2} [\bar{\phi}_{A_1 A_2}, \bar{\phi}_{B_1 B_2}] - \frac{4 g^2}{c_0^4 i_{+}^2} \bar{\phi}_{A_1 B_1} \bar{\phi}_{A_2 B_2} \\
V_{A_1 B_1 B_2 B_3} &= -\frac{g \lambda_{i\beta} \lambda_{i\gamma} F^{\beta\gamma}}{3 c_0^4 i_{+}^3} \varepsilon_{A_1 B_1 B_2 B_3}
\end{aligned}$$

Fifth Order

$$\begin{aligned}
V_{A_1 A_2 A_3 A_4 B_1} &= \frac{i \sqrt{2} g^2 i_{\pm}}{3 c_0^5 i_{-}^2 i_{+}^3} (4(i_{-} \lambda_{(i+1)\gamma} - i_{+} \lambda_{(i-1)\gamma}) \varepsilon_{A_2 A_3 A_4 C} \bar{\phi}_{A_1 B_1} \psi^{\gamma C} \\
&\quad - 6 i_{-} \lambda_{(i+1)\gamma} \psi^{\gamma C} \varepsilon_{A_1 A_2 B_1 C} \bar{\phi}_{A_3 A_4}) \\
V_{A_1 A_2 A_3 B_1 B_2} &= \frac{i \sqrt{2} g^2 \lambda_{(i+1)\beta}}{3 c_0^5 i_{+}^3} (\varepsilon_{A_2 A_3 B_1 B_2} [\bar{\phi}_{A_1 C}, \psi^{\beta C}] + \varepsilon_{A_1 A_2 A_3 C} [\bar{\phi}_{B_1 B_2}, \psi^{\beta C}] \\
&\quad - \varepsilon_{A_1 B_1 B_2 C} [\bar{\phi}_{A_2 A_3}, \psi^{\beta C}] - 4 \varepsilon_{A_1 A_2 B_1 C} \psi^{\beta C} \bar{\phi}_{A_3 B_2} \\
&\quad - 8 \varepsilon_{A_2 A_3 B_1 C} \bar{\phi}_{A_1 B_2} \psi^{\beta C}) \\
&\quad + \frac{2 i \sqrt{2} g^2 i_{\pm} \lambda_{i\gamma}}{c_0^5 i_{-} i_{+}^3} \varepsilon_{A_1 B_1 B_2 C} \psi^{\gamma C} \bar{\phi}_{A_2 A_3} \\
V_{A_1 A_2 B_1 B_2 B_3} &= \frac{2 i \sqrt{2} g^2 \lambda_{i\gamma}}{3 c_0^5 i_{+}^3} \left(- [\bar{\phi}_{A_1 C}, \psi^{\gamma C}] \varepsilon_{A_2 B_1 B_2 B_3} + 3 [\bar{\phi}_{A_1 B_1}, \psi^{\gamma C}]_{+} \varepsilon_{A_2 B_2 B_3 C} \right)
\end{aligned}$$

where $[X, Y]_{+} := XY + YX$ denotes the anticommutator.

Sixth Order

$$\begin{aligned}
V_{A_1 A_2 A_3 A_4 B_1 B_2} = & -\frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{24c_0^6i_+^4}\varepsilon_{A_1 A_2 A_3 A_4} [\bar{\phi}_{B_1 B_2}, F^{\beta\alpha}] \\
& + \frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{6c_0^6i_+^4}\varepsilon_{A_1 A_2 A_3 B_1} (F^{\beta\alpha}\bar{\phi}_{A_4 B_2} + 3\bar{\phi}_{A_4 B_2}F^{\beta\alpha}) \\
& - \frac{\sqrt{2}g^2i_{\pm}}{2c_0^6i_-i_+^4}\lambda_{i\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}\bar{\phi}_{A_1 A_4}\varepsilon_{A_2 A_3 B_1 B_2} \\
& + \frac{\sqrt{2}g^3i_{\pm}}{2c_0^6i_-i_+^3}(2[\bar{\phi}_{A_1 A_2}, \bar{\phi}_{B_1 B_2}]\bar{\phi}_{A_3 A_4} + 8\bar{\phi}_{A_2 B_1}\bar{\phi}_{A_3 B_2}\bar{\phi}_{A_1 A_4}) \\
& + \frac{g^2}{3c_0^6i_-i_+^4}(\varepsilon_{A_2 A_3 A_4 C}\varepsilon_{A_1 B_1 B_2 D}(i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta}) \\
& \quad + \varepsilon_{A_1 B_1 B_2 C}\varepsilon_{A_2 A_3 A_4 D}(i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta} + 4i_-i_{\pm}\lambda_{i\gamma}\lambda_{(i+1)\delta} \\
& \quad - 4i_+i_{\pm}\lambda_{i\gamma}\lambda_{(i-1)\delta}) \\
& \quad + \varepsilon_{A_2 A_3 B_1 C}\varepsilon_{A_1 A_4 B_2 D}(6i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta}))\psi^{\gamma C}\psi^{\delta D}
\end{aligned}$$

and

$$\begin{aligned}
V_{A_1 A_2 A_3 B_1 B_2 B_3} = & \frac{\sqrt{2}g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6i_+^4}([\bar{\phi}_{A_1 A_2}, F^{\gamma\alpha}]\varepsilon_{A_3 B_1 B_2 B_3} \\
& + 3[\bar{\phi}_{A_2 B_1}, F^{\gamma\alpha}]_+\varepsilon_{A_3 B_2 B_3 A_1} + 3\bar{\phi}_{A_1 B_3}F^{\gamma\alpha}\varepsilon_{A_2 A_3 B_1 B_2}) \\
& + \frac{\sqrt{2}g^2i_{\pm}\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}\bar{\phi}_{A_1 A_2}}{3c_0^6i_-i_+^4}\varepsilon_{A_3 B_1 B_2 B_3} \\
& + \frac{2\sqrt{2}g^3}{18c_0^6i_+^3}([\bar{\phi}_{A_2 C}, [\bar{\phi}_{A_1 D}, \bar{\phi}_{EF}]]\varepsilon_{CDEF}\varepsilon_{A_3 B_1 B_2 B_3} \\
& \quad + 6[\bar{\phi}_{A_2 B_1}, [\bar{\phi}_{A_1 A_3}, \phi_{B_2 B_3}]]_+ \\
& \quad - 6\bar{\phi}_{A_1 B_3}[\bar{\phi}_{A_2 A_3}, \bar{\phi}_{B_1 B_2}] - 24\bar{\phi}_{A_1 B_3}\bar{\phi}_{A_2 B_1}\bar{\phi}_{A_3 B_2}) \\
& + \frac{4g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6i_+^4}\left(-\varepsilon_{A_1 A_2 C D}\varepsilon_{A_3 B_1 B_2 B_3}[\psi^{\alpha D}, \psi^{\gamma C}]_+ \right. \\
& \quad + 3\varepsilon_{A_3 B_2 B_3 C}\varepsilon_{A_1 A_2 B_1 D}[\psi^{\alpha D}, \psi^{\gamma C}] \\
& \quad \left. - 3\varepsilon_{A_1 B_2 B_3 C}\varepsilon_{A_2 A_3 B_1 D}\psi^{\gamma C}\psi^{\alpha D}\right) \\
V_{A_1 A_2 B_1 B_2 B_3 B_4} = & \frac{\sqrt{2}g^2\lambda_{i\gamma}\lambda_{i\beta}[F^{\gamma\beta}, \bar{\phi}_{A_1 B_1}]_+\varepsilon_{A_2 B_2 B_3 B_4}}{3c_0^6i_+^4} \\
& + \frac{2g^2\lambda_{i\gamma}\lambda_{i\delta}\psi^{\gamma C}\psi^{\delta D}}{c_0^6i_+^4}\varepsilon_{A_1 B_3 B_4 C}\varepsilon_{A_2 B_1 B_2 D}
\end{aligned}$$

Seventh Order

$$\begin{aligned}
V_{A_1 A_2 A_3 B_1 B_2 B_3 B_4} = & \frac{2ig^2 \lambda_{i\gamma} \lambda_{i\beta} \lambda_{(i+1)\epsilon}}{9c_0^7 i_+^5} \varepsilon_{A_1 A_2 B_1 C} \varepsilon_{A_3 B_2 B_3 B_4} (2F^{\gamma\beta} \psi^{\epsilon C} + \psi^{\epsilon C} F^{\gamma\beta}) \\
& - \frac{ig^2 \lambda_{i\epsilon} \lambda_{i\beta} \lambda_{(i+1)\gamma}}{3c_0^7 i_+^5} \varepsilon_{A_1 B_1 B_2 C} \varepsilon_{A_2 A_3 B_3 B_4} (F^{\gamma\beta} \psi^{\epsilon C} + 2\psi^{\epsilon C} F^{\gamma\beta}) \\
& + \frac{8ig^3 \lambda_{i\gamma} [\bar{\phi}_{A_1 C}, \psi^{\gamma C}], \bar{\phi}_{A_2 B_1}]_+ \varepsilon_{A_3 B_2 B_3 B_4}}{9c_0^7 i_+^4} \\
& + \frac{2ig^3 \lambda_{i\gamma}}{3c_0^7 i_+^4} \varepsilon_{A_3 B_3 B_4 D} [[\bar{\phi}_{A_1 A_2}, \bar{\phi}_{B_1 B_2}], \psi^{\gamma D}]_+ \\
& + \frac{8ig^3 \lambda_{i\gamma}}{9c_0^7 i_+^4} \bar{\phi}_{A_1 B_4} \left([\bar{\phi}_{A_2 C}, \psi^{\gamma C}] \varepsilon_{A_3 B_1 B_2 B_3} - 3 [\bar{\phi}_{A_2 B_1}, \psi^{\gamma C}]_+ \varepsilon_{A_3 B_2 B_3 C} \right) \\
& + \frac{2ig^3 \lambda_{i\gamma}}{3c_0^7 i_+^4} \varepsilon_{A_1 B_3 B_4 C} \psi^{\gamma C} [\bar{\phi}_{A_2 A_3}, \bar{\phi}_{B_1 B_2}] \\
& + \frac{8ig^3 \lambda_{i\gamma}}{3c_0^7 i_+^4} \varepsilon_{A_1 B_3 B_4 C} \psi^{\gamma C} \bar{\phi}_{A_2 B_1} \bar{\phi}_{A_3 B_2}
\end{aligned}$$

and

$$V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3} = V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3} |_{\phi\phi\psi} + V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3} |_{F\psi}$$

with

$$\begin{aligned}
& V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3} |_{\phi\phi\psi} \\
& = \frac{ig^3 \lambda_{(i+1)\beta}}{9c_0^7 i_+^4} \left(4\varepsilon_{A_4 B_1 B_2 B_3} [\bar{\phi}_{A_1 A_2}, [\bar{\phi}_{A_3 C}, \psi^{\beta C}]] - 2\varepsilon_{A_3 A_4 B_2 B_3} [\bar{\phi}_{A_1 B_1}, [\bar{\phi}_{A_2 C}, \psi^{\beta C}]]_+ \right. \\
& \quad - \varepsilon_{A_4 B_1 B_2 B_3} [\psi^{\beta C}, [\bar{\phi}_{A_1 A_2}, \bar{\phi}_{A_3 C}]] - 2\varepsilon_{A_4 B_1 B_2 B_3} [\bar{\phi}_{A_3 C}, [\psi^{\beta C}, \bar{\phi}_{A_1 A_2}]] \\
& \quad + 3\varepsilon_{A_1 A_2 A_4 C} [\bar{\phi}_{A_3 B_1}, [\psi^{\beta C}, \phi_{B_2 B_3}]]_+ + 3\varepsilon_{A_1 B_2 B_3 C} [\bar{\phi}_{A_3 B_1}, [\bar{\phi}_{A_2 A_4}, \psi^{\beta C}]]_+ \\
& \quad - \varepsilon_{A_4 B_1 B_2 B_3} [\psi^{\beta C}, [\bar{\phi}_{A_1 A_2}, \bar{\phi}_{A_3 C}]] - 9\varepsilon_{A_2 A_3 B_1 C} [\psi^{\beta C}, [\bar{\phi}_{A_1 A_4}, \bar{\phi}_{B_2 B_3}]]_+ \\
& \quad - 12\varepsilon_{A_1 A_2 B_3 C} \psi^{\beta C} \bar{\phi}_{A_3 B_1} \bar{\phi}_{A_4 B_2} \\
& \quad + \bar{\phi}_{A_1 B_3} (6\varepsilon_{A_2 A_3 A_4 C} [\psi^{\beta C}, \bar{\phi}_{B_1 B_2}] + 6\varepsilon_{A_2 B_1 B_2 C} [\bar{\phi}_{A_3 A_4}, \psi^{\beta C}] \\
& \quad + 24\varepsilon_{A_2 A_3 B_1 C} [\psi^{\beta C}, \bar{\phi}_{A_4 B_2}]_+ + 12\varepsilon_{A_3 A_4 B_1 C} \bar{\phi}_{A_2 B_2} \psi^{\beta C} \\
& \quad \left. - 5\varepsilon_{A_3 A_4 B_1 B_2} [\bar{\phi}_{A_2 C}, \psi^{\beta C}]) \right) \\
& + \frac{4ig^3 i_{\pm} \lambda_{i\gamma}}{3c_0^7 i_+^4} \left(\varepsilon_{A_4 B_1 B_2 B_3} [\bar{\phi}_{A_1 C}, \psi^{\gamma C}] \bar{\phi}_{A_2 A_3} - 3\varepsilon_{A_2 B_1 B_2 C} [\bar{\phi}_{A_1 B_3}, \psi^{\gamma C}]_+ \bar{\phi}_{A_3 A_4} \right)
\end{aligned}$$

and

$$\begin{aligned}
& V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3} \big|_{F\psi} \\
&= \frac{ig^2 \lambda_{(i+1)\alpha} \lambda_{i\gamma} \lambda_{(i+1)\beta}}{18c_0^7 i_+^5} \left(9\varepsilon_{A_1 A_2 B_1 C} \varepsilon_{A_3 A_4 B_2 B_3} [\psi^{\beta C}, F^{\gamma\alpha}]_+ + 6\varepsilon_{A_2 B_2 B_3 C} \varepsilon_{A_3 A_4 B_1 A_1} \psi^{\gamma C} F^{\beta\alpha} \right. \\
&\quad \left. + \varepsilon_{A_1 A_2 A_3 C} \varepsilon_{A_4 B_1 B_2 B_3} [F^{\beta\alpha}, \psi^{\gamma C}] + 3\varepsilon_{A_4 B_2 B_3 C} \varepsilon_{A_2 A_3 B_1 A_1} [F^{\beta\alpha}, \psi^{\gamma C}]_+ \right) \\
&\quad + \frac{2ig^2 i_{\pm} \lambda_{(i+1)\delta} \lambda_{i\gamma} \lambda_{i\beta}}{9c_0^7 i_- i_+^5} \varepsilon_{A_1 A_2 A_3 C} \varepsilon_{A_4 B_1 B_2 B_3} F^{\gamma\beta} \psi^{\delta C} \\
&\quad + \frac{2ig^2 i_{\pm} \lambda_{i\beta} \lambda_{i\gamma} \lambda_{(i-1)\delta}}{9c_0^7 i_-^2 i_+^4} \varepsilon_{A_1 B_1 B_2 B_3} \varepsilon_{A_2 A_3 A_4 C} F^{\beta\gamma} \psi^{\delta C}
\end{aligned}$$

Eighth Order

$$\begin{aligned}
V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4} &= V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4} \big|_{\phi^4} + V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4} \big|_{F\phi\phi} \\
&\quad + V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4} \big|_{\phi\psi\psi} + V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4} \big|_{FF}
\end{aligned}$$

with

$$\begin{aligned}
V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4} \big|_{\phi^4} &= \frac{g^4}{18c_0^8 i_+^4} \left(-2\varepsilon_{DEFG} \varepsilon_{A_4 B_2 B_3 B_4} \left[[\bar{\phi}_{A_2 D}, [\bar{\phi}_{A_1 E}, \bar{\phi}_{FG}]]_-, \bar{\phi}_{A_3 B_1} \right]_+ \right. \\
&\quad + 3 \left[[\bar{\phi}_{A_2 A_3}, \bar{\phi}_{B_1 B_2}], [\bar{\phi}_{A_1 A_4}, \bar{\phi}_{B_3 B_4}] \right]_+ \\
&\quad - 2\bar{\phi}_{A_2 B_4} [\bar{\phi}_{A_3 D}, [\bar{\phi}_{A_1 E}, \bar{\phi}_{FG}]] \varepsilon_{A_4 B_1 B_2 B_3} \varepsilon_{DEFG} \\
&\quad - 12\bar{\phi}_{A_2 B_4} [\bar{\phi}_{A_3 B_1}, [\bar{\phi}_{A_1 A_4}, \bar{\phi}_{B_2 B_3}]]_+ \\
&\quad + 3 [\bar{\phi}_{A_1 A_2}, \bar{\phi}_{B_3 B_4}] [\bar{\phi}_{A_3 A_4}, \bar{\phi}_{B_1 B_2}] \\
&\quad + 12 [\bar{\phi}_{A_1 A_2}, \bar{\phi}_{B_3 B_4}] \bar{\phi}_{A_3 B_1} \bar{\phi}_{A_4 B_2} \\
&\quad + 2\bar{\phi}_{A_1 B_4} [\bar{\phi}_{A_3 C}, [\bar{\phi}_{A_2 D}, \bar{\phi}_{EF}]] \varepsilon_{CDEF} \varepsilon_{A_4 B_1 B_2 B_3} \\
&\quad + 12\bar{\phi}_{A_1 B_4} [\bar{\phi}_{A_3 B_1}, [\bar{\phi}_{A_2 A_4}, \bar{\phi}_{B_2 B_3}]]_+ \\
&\quad - 12\bar{\phi}_{A_1 B_4} \bar{\phi}_{A_2 B_3} [\bar{\phi}_{A_3 A_4}, \bar{\phi}_{B_1 B_2}] \\
&\quad \left. - 48\bar{\phi}_{A_1 B_4} \bar{\phi}_{A_2 B_3} \bar{\phi}_{A_3 B_1} \bar{\phi}_{A_4 B_2} \right)
\end{aligned}$$

and

$$\begin{aligned}
& V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4} |F\phi\phi \\
&= \frac{g^3 \lambda_{i\beta} \lambda_{(i+1)\gamma}}{36 c_0^8 i_+^5} \left(-9 \varepsilon_{A_3 A_4 B_3 B_4} [F^{\gamma\beta}, [\bar{\phi}_{A_1 A_2}, \bar{\phi}_{B_1 B_2}]]_+ \right. \\
&\quad + 4 \varepsilon_{A_4 B_2 B_3 B_4} \left[[\bar{\phi}_{A_2 A_1}, F^{\gamma\beta}]_-, \bar{\phi}_{A_3 B_1} \right]_+ \\
&\quad + 8 \bar{\phi}_{A_1 B_4} \left([\bar{\phi}_{A_2 A_3}, F^{\gamma\beta}] \varepsilon_{A_4 B_1 B_2 B_3} + 3 [\bar{\phi}_{A_2 B_1}, F^{\gamma\beta}]_+ \varepsilon_{A_3 B_2 B_3 A_4} \right) \\
&\quad \left. - 12 \varepsilon_{A_1 A_2 B_1 B_2} [F^{\gamma\beta}, \bar{\phi}_{A_3 B_3} \bar{\phi}_{A_4 B_4}]_+ \right) \\
&\quad - \frac{2g^3 i_{\pm} \lambda_{i\gamma} \lambda_{i\beta}}{3 c_0^8 i_- i_+^5} \varepsilon_{A_1 B_1 B_2 B_3} [F^{\gamma\beta}, \bar{\phi}_{A_2 B_4}]_+ \bar{\phi}_{A_3 A_4}
\end{aligned}$$

and

$$\begin{aligned}
& V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4} |\phi\psi\psi \\
&= \frac{\sqrt{2} g^3 \lambda_{i\gamma} \lambda_{(i+1)\beta}}{18 c_0^8 i_+^5} \left(8 \varepsilon_{A_2 A_3 B_1 D} \varepsilon_{A_4 B_2 B_3 B_4} [[\bar{\phi}_{A_1 C}, \psi^{\gamma C}], \psi^{\beta D}]_- \right. \\
&\quad - 12 \varepsilon_{A_2 A_3 B_4 D} \varepsilon_{A_4 B_2 B_3 C} \psi^{\beta D} [\bar{\phi}_{A_1 B_1}, \psi^{\gamma C}]_+ \\
&\quad - 3 \varepsilon_{A_2 B_1 B_2 C} \varepsilon_{A_3 A_4 B_3 B_4} ([\bar{\phi}_{A_1 D}, \psi^{\beta D}] \psi^{\gamma C} - 2 \psi^{\gamma C} [\bar{\phi}_{A_1 D}, \psi^{\beta D}]) \\
&\quad + 4 \varepsilon_{A_4 B_2 B_3 B_4} \left(\varepsilon_{A_1 A_2 C D} [[\psi^{\beta D}, \psi^{\gamma C}]_+, \bar{\phi}_{A_3 B_1}]_+ \right. \\
&\quad \left. + \varepsilon_{A_1 A_3 B_1 D} [\psi^{\beta D}, [\bar{\phi}_{A_2 C}, \psi^{\gamma C}]]_- \right) \\
&\quad + 3 \varepsilon_{A_4 B_3 B_4 C} \left(\varepsilon_{A_1 A_2 A_3 D} [[\psi^{\beta D}, \bar{\phi}_{B_1 B_2}], \psi^{\gamma C}]_- \right. \\
&\quad \left. + \varepsilon_{A_1 B_1 B_2 D} [[\bar{\phi}_{A_2 A_3}, \psi^{\beta D}], \psi^{\gamma C}]_- \right) \\
&\quad + 4 \bar{\phi}_{A_1 B_4} \left(-2 \varepsilon_{A_2 A_3 C D} \varepsilon_{A_4 B_1 B_2 B_3} [\psi^{\beta D}, \psi^{\gamma C}]_+ \right. \\
&\quad + 6 \varepsilon_{A_4 B_2 B_3 C} \varepsilon_{A_2 A_3 B_1 D} [\psi^{\beta D}, \psi^{\gamma C}]_- \\
&\quad \left. - 3 \varepsilon_{A_2 B_2 B_3 C} \varepsilon_{A_3 A_4 B_1 D} \psi^{\gamma C} \psi^{\beta D} \right) \\
&\quad + 3 \varepsilon_{A_1 B_3 B_4 C} \psi^{\gamma C} (2 \varepsilon_{A_2 A_3 A_4 D} [\psi^{\beta D}, \bar{\phi}_{B_1 B_2}] + 2 \varepsilon_{A_2 B_1 B_2 D} [\bar{\phi}_{A_3 A_4}, \psi^{\beta D}] \\
&\quad - \varepsilon_{A_2 A_3 B_1 B_2} [\bar{\phi}_{A_4 D}, \psi^{\beta D}] + 8 \varepsilon_{A_2 A_3 B_1 D} [\bar{\phi}_{A_4 B_2}, \psi^{\beta D}]_+ \\
&\quad \left. + 4 \varepsilon_{A_2 A_3 B_1 D} \bar{\phi}_{A_4 B_2} \psi^{\beta D} \right) \\
&\quad - \frac{2\sqrt{2} g^3 i_{\pm} \lambda_{i\beta} \lambda_{i\gamma}}{c_0^8 i_- i_+^5} \varepsilon_{A_1 B_3 B_4 C} \varepsilon_{A_2 B_1 B_2 D} \psi^{\gamma C} \psi^{\beta D} \bar{\phi}_{A_3 A_4}
\end{aligned}$$

and

$$V_{A_1 A_2 A_3 A_4 B_1 B_2 B_3 B_4}|_{FF} = -\frac{g^2 \lambda_{i\gamma} \lambda_{i\beta} \lambda_{(i+1)\epsilon} \lambda_{(i+1)\alpha}}{36 c_0^8 i_+^6} \varepsilon_{A_1 A_2 A_3 B_1} \varepsilon_{A_4 B_2 B_3 B_4} (3 F^{\gamma\beta} F^{\epsilon\alpha} + F^{\epsilon\alpha} F^{\gamma\beta}) \\ + \frac{g^2 \lambda_{i\epsilon} \lambda_{i\beta} \lambda_{(i+1)\gamma} \lambda_{(i+1)\alpha}}{8 c_0^8 i_+^6} \varepsilon_{A_1 A_2 B_1 B_2} \varepsilon_{A_3 A_4 B_3 B_4} F^{\gamma\beta} F^{\epsilon\alpha}$$

The General Structure

From the above formulas for $\mathcal{V}_{i,i+1}$, we see that higher order terms factor into terms with the structure of lower order terms:

$$\mathcal{V}_{i,i+1} \sim \sum \Pi \left(1 + \bar{\phi} \cdot \eta^2 + \frac{1}{\sqrt{g}} \psi \cdot \eta^3 + \frac{1}{g} F \cdot \eta^4 \right)$$

It is understood that this is not an equation but only a similarity which helps memorise the types of terms occurring.

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